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An improved lower bound and approximation algorithm for binary constrained quadratic programming problem

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Abstract This paper presents an improved lower bound and an approximation algorithm based on spectral decomposition for the binary constrained quadratic programming problem. To decompose spectrally the quadratic matrix in the objective function, we construct a low rank problem that provides a lower bound. Then an approximation algorithm for the binary quadratic programming problem together with a worst case performance analysis for the algorithm is provided.

Keywords Quadratic integer programming · Lower bound · Approximation algorithm · Spectral decomposition

1 Introduction

In this paper the following binary quadratic integer programming problem

$$\begin{aligned} \gamma = \min \quad & x^T Qx \\ \text{s.t.} \quad & x \in \{-1, 1\}^n \end{aligned} \tag{QIP}$$

is considered, where Q is an $n \times n$ real symmetric matrix.

(QIP) is a classic NP-Hard problem [9] and is not possible to be solved in polynomially computational time unless P=NP. It has been intensively studied because of its theoretical importance and wide applications, for example, the classic Max-Cut is a special case of the problem.

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For the binary quadratic programming, an important issue is to compute a good lower bound. A well-performed lower bound is important for a branch-and-bound algorithm to solve it, and a better lower bound may then provide a shorter computation time in algorithms [19]. For some problems in applications, obtaining a well-performed lower bound in acceptable time is enough [11–14,16]. Therefore the lower bound problem is important.

Allemand et al. [1] proposed a polynomial time algorithm to solve a special sub-classes of (QIP) such that $Q \preceq 0$ and $Rank(Q) = k$ for some fixed k. Their algorithm is based on extreme points enumeration on zonotope in low dimension, with time complexity $\mathcal{O}(n^k)$ for $k = 1, 2$ and $\mathcal{O}(n^{k-1})$ for $k > 2$. To extend this result, Halikias et al. [15] presented a new method to obtain a lower bound without the rank restriction on Q, which becomes tighter as the computation time increases. In their method, they formulated a low rank problem using spectral decomposition, and by solving this low rank problem, a lower bound of (QIP) is obtained. They also suggested using a semidefinite relaxation method to reformulate the original (QIP) problem at the first step, then using the low rank decomposition method to compute lower bounds. Using this reformulation strategy, each newly obtained lower bound is always no worse than its former.

Halikias et al. [15] dropped a non-negative tail term produced in the spectral decomposition directly. However, we discover that to underestimate this non-negative tail term sometimes can improve their lower bound. In this paper, an improved lower bound method is adopted by considering the tail term.

Another important issue for NP-hard combinatorial optimization problems, including (QIP), is to design polynomial approximation algorithms to obtain acceptable solutions. To obtain a good approximation solution for (QIP) is not easy for general cases. However, for some special cases, there exist some algorithms with good approximation ratios. For example the randomized algorithm based on semidefinite relaxation for the Max-Cut provides a solution with an approximation ratio of 0.878 [10].

Our improved lower bound method can also work as an approximation algorithm for (QIP), which performs well for some cases. We give a worst case ratio for the algorithm.

This paper is arranged as follows. Some simplified proofs for results in [15] are presented in Sect. 2. A lower bound method considering the tail term is studied in Sect. 3 and an approximation algorithm and its worst case performance are presented in Sect. 4.

2 Spectral decomposition

For (QIP), denote the eigenvalues of Q by $\lambda_1, \lambda_2, \ldots, \lambda_n$, with their corresponding eigenvectors $v_1, v_2, \ldots, v_n$. We also assume that $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$ and the eigenvectors are unitary and mutually orthogonal. Then the spectral decomposition of Q becomes

$$Q = \sum_{i=1}^{n} \lambda_i v_i v_i^T, \tag{1}$$

and the eigenvectors $v_1, v_2, \ldots, v_n$ satisfy that

$$\sum_{i=1}^{n} v_i v_i^T = I, \tag{2}$$

where I is the identity matrix. Thus for any vectors $x \in \{-1, 1\}^n$, we have

$$\sum_{i=1}^{n} x^T v_i v_i^T x = n. \tag{3}$$

For any index set $B \subseteq \{1, 2, \ldots, n\}$ and $B^c = \{1, 2, \ldots, n\} - B$, we have

$$\sum_{i \in B} x^T v_i v_i^T x = n - \sum_{i \in B^c} x^T v_i v_i^T x. \tag{4}$$

Then (QIP) becomes

$$\begin{aligned} x^T Qx &= \sum_{i=1}^{n} \lambda_i x^T v_i v_i^T x \\ &= \sum_{i=1}^{k} \lambda_i x^T v_i v_i^T x + \sum_{i=k+1}^{n} \lambda_i x^T v_i v_i^T x \\ &\geqslant \sum_{i=1}^{k} \lambda_i x^T v_i v_i^T x + \lambda_{k+1} \sum_{i=k+1}^{n} x^T v_i v_i^T x \\ &= \sum_{i=1}^{k} \lambda_i x^T v_i v_i^T x + \lambda_{k+1} \left(n - \sum_{i=1}^{k} x^T v_i v_i^T x \right) \\ &= \sum_{i=1}^{k} (\lambda_i - \lambda_{k+1}) x^T v_i v_i^T x + n\lambda_{k+1}. \end{aligned} \tag{5}$$

Now define a sequence of optimization problems

$$\gamma_k = \min_{x \in \{-1,1\}^n} x^T P_k x + n\lambda_{k+1}, \tag{$\mathcal{P}_k$}$$

where

$$P_k = \sum_{i=1}^{k} (\lambda_i - \lambda_{k+1}) v_i v_i^T, \tag{6}$$

for $k = 0, 1, \ldots, n-1$. In the following, we always assume that $k \in \{0, 1, \ldots, n-1\}$ if no specification is given.

The relationship between γ and γ_k is stated as follows.

Lemma 1 $\gamma_k \leqslant \gamma$.

Proof We have $x^T Qx \geqslant x^T P_k x + n\lambda_{k+1}$ from (5) for any $x \in \{-1, 1\}^n$. Hence

$$\gamma = \min_{x \in \{-1,1\}^n} x^T Qx \geqslant \min_{x \in \{-1,1\}^n} x^T P_k x + n\lambda_{k+1} = \gamma_k.$$

□

Thus γ_k is a lower bound of (QIP). Meanwhile, $P_k = \sum_{i=1}^{k} (\lambda_i - \lambda_{k+1}) v_i v_i^T \preceq 0$, since $\lambda_i \leqslant \lambda_{k+1}$ for $i = 1, 2, \ldots, k$.

Lemma 2 [18] *γ_k could be computed in $\mathcal{O}(n^k)$ computation time for $k = 1, 2$ and in $\mathcal{O}(n^{k-1})$ computation time for $k > 2$.*

The number of all extreme points of the zonotope $\mathcal{Z} = \{V^T x | x \in [-1, 1]^n\}$, where $V = [v_1, v_2, \ldots, v_k]$, is $\mathcal{O}(n^k)$ for $k = 1, 2$ and $\mathcal{O}(n^{k-1})$ for $k > 2$, respectively, as shown in [3]. The algorithm in [18] to compute γ_k is based on the *reverse-enumeration* of all extreme points of $\mathcal{Z}$ [2,5]. So for a given k, we obtain γ_k in $\mathcal{O}(n^k)$ for $k = 1, 2$ and $\mathcal{O}(n^{k-1})$ for $k > 2$, this computation time is acceptable for small k.

Remark It is obvious that the computation of the optimal γ requires at most 2^n comparisons of the objective function values, while Lemma 2 states the polynomial complexity $\mathcal{O}(n^{k-1})$ for $k > 2$ to compute γ_k. Actually in computation, evaluating γ_k could be more expensive than evaluating γ for a large k.

The bound γ_k is equivalent to that defined in Corollary 3.2 in [15]. In the remaining part of this section, Theorem 1–3 will present properties of γ_k, which are similar to the results of Lemma 3.1 and Corollary 3.2 in [15]. Our new simple proofs are provided here. Furthermore, the following theorem shows that the lower bound γ_k could be improved by increasing k.

Theorem 1 [15] $\gamma_0 \leqslant \gamma_1 \leqslant \cdots \leqslant \gamma_{n-1} = \gamma$.

Proof For any $0 \leqslant k \leqslant n - 2$, let $\hat{x}$ be the optimal solution of $\min_{x\in\{-1,1\}^n} x^T P_{k+1} x$, then we have

$$\gamma_k \leqslant \hat{x}^T P_k \hat{x} + n\lambda_{k+1} = \sum_{i=1}^{k} (\lambda_i - \lambda_{k+1}) \hat{x}^T v_i v_i^T \hat{x} + n\lambda_{k+1}, \tag{7}$$

$$\gamma_{k+1} = \hat{x}^T P_{k+1} \hat{x} + n\lambda_{k+2} = \sum_{i=1}^{k+1} (\lambda_i - \lambda_{k+2}) \hat{x}^T v_i v_i^T \hat{x} + n\lambda_{k+2}. \tag{8}$$

From (7) and (8), we have

$$\begin{aligned} \gamma_k - \gamma_{k+1} &\leqslant n(\lambda_{k+1} - \lambda_{k+2}) - (\lambda_{k+1} - \lambda_{k+2}) \sum_{i=1}^{k+1} \hat{x}^T v_i v_i^T \hat{x} \\ &= (\lambda_{k+1} - \lambda_{k+2}) \left(n - \sum_{i=1}^{k+1} \hat{x}^T v_i v_i^T \hat{x} \right) \\ &\leqslant 0, \end{aligned} \tag{9}$$

thus $\gamma_0 \leqslant \gamma_1 \leqslant \cdots \leqslant \gamma_{n-1}$ and

$$\begin{aligned} \gamma_{n-1} &= \min_{x\in\{-1,1\}^n} \sum_{i=1}^{n-1} (\lambda_i - \lambda_n) x^T v_i v_i^T x + n\lambda_n \\ &= \min_{x\in\{-1,1\}^n} x^T \left(\sum_{i=1}^{n} \lambda_i v_i v_i^T - \lambda_n I \right) x + n\lambda_n \\ &= \min_{x\in\{-1,1\}^n} x^T Q x = \gamma. \end{aligned}$$

□

Theorem 1 shows that γ_k gets tighter as k increases (with more computation cost). The following two theorems will further present more properties of γ_k.

Theorem 2 [15] *Denote $\hat{x}$ as an optimal solution of $(\mathcal{P}_k)$. If $\sum_{i=1}^{k+1} \hat{x}^T v_i v_i^T \hat{x} = n$, then $\gamma_k = \gamma$ and $\hat{x}$ is an optimal solution of (QIP).*

Proof It is easy to verify that

$$\hat{x}^T Q\hat{x} = \sum_{i=1}^{n} \lambda_i \hat{x}^T v_i v_i^T \hat{x} = \sum_{i=1}^{k} (\lambda_i - \lambda_{k+1})\hat{x}^T v_i v_i^T \hat{x} + n\lambda_{k+1} + \sum_{i=k+2}^{n} (\lambda_i - \lambda_{k+1})\hat{x}^T v_i v_i^T \hat{x}.$$

If $\sum_{i=1}^{k+1} \hat{x}^T v_i v_i^T \hat{x} = n$, then $\sum_{i=k+2}^{n} \hat{x}^T v_i v_i^T \hat{x} = 0$. Then for any $i \geqslant k+2$, $\hat{x}^T v_i v_i^T \hat{x} = 0$. So $\sum_{i=k+2}^{n} (\lambda_i - \lambda_{k+1}) x^T v_i v_i^T x = 0$ and $\hat{x}^T Q\hat{x} = \sum_{i=1}^{k} (\lambda_i - \lambda_{k+1})\hat{x}^T v_i v_i^T \hat{x} + n\lambda_{k+1} = \gamma_k$. Thus $\hat{x}$ is also an optimal solution of (QIP) and $\gamma_k = \gamma$. □

Theorem 3 [15] *For any $0 \leqslant k \leqslant n-2$, if $\lambda_{k+1} = \lambda_{k+2}$, then $\gamma_k = \gamma_{k+1}$. If $\lambda_{k+1} < \lambda_{k+2}$ and $\gamma_k = \gamma_{k+1}$, then $\gamma_k = \gamma$ and an optimal solution $\hat{x}$ of $(\mathcal{P}_k)$ is also an optimal solution of (QIP).*

Proof If $\lambda_{k+1} = \lambda_{k+2}$, we can easily verify that $P_k = P_{k+1}$. So γ_k and γ_{k+1} have the same objective function, and thus $\gamma_k = \gamma_{k+1}$. If $\lambda_{k+1} < \lambda_{k+2}$ and $\gamma_k = \gamma_{k+1}$, from (9), we have

$$0 = \gamma_k - \gamma_{k+1} \leqslant (\lambda_{k+1} - \lambda_{k+2}) \left(n - \sum_{i=1}^{k+1} \hat{x}^T v_i v_i^T \hat{x} \right) \leqslant 0, \tag{10}$$

and

$$(\lambda_{k+1} - \lambda_{k+2}) \left(n - \sum_{i=1}^{k+1} \hat{x}^T v_i v_i^T \hat{x} \right) = 0. \tag{11}$$

By the assumption that $\lambda_{k+1} < \lambda_{k+2}$ and (11), we have

$$\sum_{i=1}^{k+1} \hat{x}^T v_i v_i^T \hat{x} = n. \tag{12}$$

From Theorem 2, we know $\gamma_k = \gamma$ and $\hat{x}$ is an optimal solution of (QIP). □

The above properties present some sufficient conditions for zero gap between γ_k and γ. Specially, when $\lambda_{k+1} = \lambda_k = \cdots = \lambda_m$, where $k+1 < m \leqslant n$, we only need to compute γ_k rather than γ_s with $k < s \leqslant m-1$ to obtain the same lower bound.

3 Improved lower bounds

Considering the decomposed form

$$x^T Qx = \sum_{i=1}^{n} \lambda_i x^T v_i v_i^T x = \sum_{i=1}^{k} (\lambda_i - \lambda_{k+1}) x^T v_i v_i^T x + n\lambda_{k+1} + \sum_{i=k+2}^{n} (\lambda_i - \lambda_{k+1}) x^T v_i v_i^T x,$$

for $k = 0, 1, \ldots, n-2$, we define

$$R_k = \sum_{i=k+2}^{n} (\lambda_i - \lambda_{k+1}) v_i v_i^T, \tag{13}$$

and a binary constrained quadratic programming

$$t_k = \min_{x\in\{-1,1\}^n} x^T R_k x. \qquad (\mathcal{R}_k)$$

Remark For any given k, $\mathcal{R}_k = Q - \sum_{i=1}^{k}(\lambda_i - \lambda_{k+1})v_i v_i^T$ when the first $k+1$ eigenvalues and the first k eigenvectors are given.

If $k = n-1$, we directly define $t_k = 0$ and R_k is a zero $n \times n$ matrix. Thus we have

$$x^T Qx = x^T P_k x + n\lambda_{k+1} + x^T R_k x. \qquad (14)$$

Lemma 3 $\gamma \geqslant \gamma_k + t_k$.

Proof From the decomposition form (14), we have

$$\begin{aligned}\gamma &= \min_{x\in\{-1,1\}^n} x^T Qx\\ &= \min_{x\in\{-1,1\}^n} \{x^T P_k x + n\lambda_{k+1} + x^T R_k x\}\\ &\geqslant \min_{x\in\{-1,1\}^n} \{x^T P_k x\} + n\lambda_{k+1} + \min_{x\in\{-1,1\}^n} \{x^T R_k x\}\\ &= \gamma_k + t_k.\end{aligned}$$

Thus the conclusion follows. □

$R_k = \sum_{i=k+2}^{n}(\lambda_i - \lambda_{k+1})v_i v_i^T \succeq 0$ since $\lambda_i - \lambda_{k+1} \geqslant 0$ for $i \geqslant k+2$. Thus $t_k = \min_{x\in\{-1,1\}^n} x^T R_k x \geqslant 0$.

Recall that the evaluation of lower bound γ_k in Sect. 2 directly discards t_k, which may cause an unexpected gap when $t_k > 0$. So we consider to underestimate the term t_k to improve the original lower bound γ_k.

$(\mathcal{R}_k)$ is equivalent to:

$$\begin{aligned} t_k = \min \quad & x^T R_k x\\ \text{s.t.} \quad & x_i^2 = 1 \text{ for } i = 1, 2, \ldots, n,\end{aligned}$$

and it is hard to compute t_k efficiently for general cases, thus we estimate a lower bound of t_k. The classical Lagrangian duality method is adopted to give a lower bound.

The Lagrangian function is defined as

$$L_k(x, \mu) = x^T R_k x + \sum_{i=1}^{n} \mu_i (x_i^2 - 1).$$

Its dual function is denoted as $P_k^d(\mu) = \min_{x\in\mathbb{R}^n} L_k(x, \mu)$. The Lagrangian dual problem is thus defined as

$$d_k = \max_{\mu\in S} P_k^d(\mu), \qquad (\mathcal{DR}_k)$$

where $S = \{\mu | \min_{x\in\mathbb{R}^n} L_k(x, \mu) > -\infty\}$. We have the following results.

Lemma 4 (Weak duality theorem) $d_k \leqslant t_k$.

Lemma 4 is derived from the weak duality theorem in Lagrangian duality theory. So d_k is a lower bound of t_k. Meanwhile, d_k also satisfies the following property.

Lemma 5 $d_k \geqslant 0$.

Proof Since $R_k \succeq 0$, we have $d_k = \max_{\mu \in S} P_k^d(\mu) \geqslant P_k^d(0) = \min_{x \in \mathbb{R}^n} L_k(x, 0) = \min_{x \in \mathbb{R}^n} x^T R_k x = 0$. □

Following from Lemmas 3–5, we have

Theorem 4 $\gamma_k \leqslant \gamma_k + d_k \leqslant \gamma_k + t_k \leqslant \gamma$.

This theorem shows that $\gamma_k + d_k$ is an improved lower bound of γ_k. As the following theorem shows, the computation of d_k is very efficient, i.e., d_k could be computed efficiently (polynomial time with positive precision) by solving a semidefinite problem (SDP).

Theorem 5 [17] *The Lagrangian dual problem* $(\mathcal{DR}_k)$ *is equivalent to the following SDP problem:*

$$\begin{aligned} \max \quad & \mu_1 + \mu_2 + \cdots + \mu_n \\ \text{s.t.} \quad & R_k - \begin{bmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_n \end{bmatrix} \succeq 0. \end{aligned} \tag{SDP}$$

Several duality approaches can be adopted to get lower bounds for d_k. Recently the canonical duality theory has attracted attention on applications in the binary quadratic programming. Its dual function is analytically expressed and is sometimes powerful to give the exact solutions of the programs [4,7,20]. The canonical duality theory is based on the pioneer work of Gao and Strang [6,8] and has many successful applications on nonconvex optimization problems. For the binary quadratic integer programming problem studied in this paper, both the Lagrangian dual $(\mathcal{DR}_k)$ and the SDP problem are special forms of the general canonical dual (see equation (31) and Theorem 3 in [6]), and Theorem 5 presented above can be implied from Theorem 5 proposed in [8]. However we use the classic Lagrangian duality method to show these results so as to be more accessible for readers.

The following example shows the efficiency of the new lower bound $\gamma_k + d_k$.

Example 1 $\min_{x \in \{-1,1\}^n} \sum_{k=1}^n k x_k^2$.

It is easy to verify that any point $x \in \{-1, 1\}^n$ is an optimal solution of this problem, and the optimal value is $\frac{n(n+1)}{2}$. Now computing the lower bound γ_k from $\min_{x \in \{-1,1\}^n} x^T P_k x + n\lambda_{k+1}$, where $x^T P_k x = \sum_{i=1}^k (i - k - 1)x_i^2$, we have $\gamma_k = \frac{(2n-k)(k+1)}{2}$. Next computing d_k by solving the dual problem of $\min_{x \in \{-1,1\}^n} x^T R_k x$, we have $d_k = \frac{(n-k)(n-k-1)}{2}$. The lower bound $\gamma_k + d_k$ is a tight lower bound for (QIP) (see Fig. 1 for the case $n = 10$).

Halikias et al. also suggested incorporating the semideifnite relaxation (SDR) method to reformulate the original problem (QIP), and then the lower bounds obtained from the reformulated problem by the low rank decomposition method can be further improved. More details can be found in [15]. Our method is to obtain γ_k without SDR procedure and consider the tail term t_k to improve the lower bound for (QIP). In this paper, we use a SDP formulation as Theorem 5 states to underestimate t_k. Furthermore, many lower bound methods can be adopted to underestimate the tail term in our approach. So our bounds have potential to be further improved.

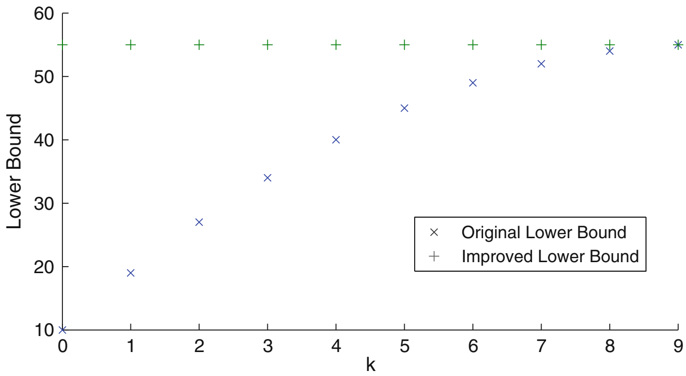

Fig. 1 Original lower bound γ_k and improved lower bound $\gamma_k + d_k$ for $k = 0, 1, \ldots, 9$

4 Approximation algorithm

As shown in Sect. 2, the low rank problem $(\mathcal{P}_k)$ provides a lower bound for (QIP). The question arises whether the optimal solution of $(\mathcal{P}_k)$ is also a good approximation solution for (QIP). In this section, we show that the solution of $(\mathcal{P}_k)$ performs well for some special cases.

Denote $\tilde{x}$ as the optimal solution of $(\mathcal{P}_k)$, and denote $\hat{x}$ as the optimal solution of (QIP). Then

$$\tilde{x}^T Q\tilde{x} = \tilde{x}^T P_k\tilde{x} + n\lambda_{k+1} + \tilde{x}^T R_k\tilde{x}, \tag{15}$$

$$\hat{x}^T Q\hat{x} = \hat{x}^T P_k\hat{x} + n\lambda_{k+1} + \hat{x}^T R_k\hat{x}. \tag{16}$$

From (15) and (16), we have

Lemma 6 $\tilde{x}^T Q\tilde{x} - \gamma \leqslant \tilde{x}^T R_k\tilde{x} - d_k$, *where* d_k *is the optimal value of problem* $(\mathcal{DR}_k)$.

Proof Since $\tilde{x}$ is the optimal solution of $(\mathcal{P}_k)$, we have

$$\tilde{x}^T P_k\tilde{x} \leqslant \hat{x}^T P_k\hat{x}. \tag{17}$$

From (15–17), we get

$$\tilde{x}^T Q\tilde{x} - \hat{x}^T Q\hat{x} \leqslant \tilde{x}^T R_k\tilde{x} - \hat{x}^T R_k\hat{x}. \tag{18}$$

Together with $\hat{x}^T R_k\hat{x} \geqslant t_k \geqslant d_k$ in Lemma 4, we then have

$$\tilde{x}^T Q\tilde{x} - \hat{x}^T Q\hat{x} \leqslant \tilde{x}^T R_k\tilde{x} - d_k. \tag{19}$$

This ends the proof. □

Let $\Delta_k = \tilde{x}^T R_k\tilde{x} - d_k$. We conclude from Lemma 6 that an approximation solution $\tilde{x}$ for (QIP) is obtained by solving $(\mathcal{P}_k)$ and a computational error bound Δ_k is then obtained by additionally solving $(\mathcal{DR}_k)$.

As Δ_k depends on the computational results, Lemma 6 does not provide a theoretical bound. The following theorem extends this result.

Theorem 6 *The error bound of the approximation solution* $\tilde{x}$ *satisfies that* $\Delta_k \leqslant n(\lambda_n - \lambda_{k+1})$.

Proof Considering the spectral decomposition form in (13) and since $d_k \geqslant 0$, we have

$$\begin{aligned}\Delta_k &\leqslant \sum_{i=k+2}^{n} (\lambda_i - \lambda_{k+1})\tilde{x}^T v_i v_i^T \tilde{x}\\ &\leqslant (\lambda_n - \lambda_{k+1}) \sum_{i=k+2}^{n} \tilde{x}^T v_i v_i^T \tilde{x}\\ &= (\lambda_n - \lambda_{k+1})\left(n - \sum_{i=1}^{k+1} \tilde{x}^T v_i v_i^T \tilde{x}\right)\\ &\leqslant n(\lambda_n - \lambda_{k+1}).\end{aligned}$$

□

This theorem gives a worst case bound for the gap between the approximation and the optimal value.

Furthermore, without loss of generality, we can assume Q is positive definite since $x^T Qx = x^T(Q + \lambda I)x - n\lambda$ for any $\lambda \in \mathbb{R}$ and $x \in \{0, 1\}^n$ and a large enough λ would be chosen such that $Q + \lambda I$ is positive definite. The assumption of the positive definite matrix ensures the following result.

Theorem 7 *For (QIP) problem with a positive definite matrix Q, let* V^* *be the optimal value of (QIP), and* V^k *be the objective value obtained by solving* $(\mathcal{P}_k)$*, then* $\frac{V^k}{V^*} \leqslant \frac{\lambda_n}{\lambda_{k+1}}$.

Proof Denote $\tilde{x}$ as an optimal solution of $(\mathcal{P}_k)$ and $z_i = v_i^T \tilde{x}$ for $i = 1, 2, \ldots, n$, then we have

$$V^k = \sum_{i=1}^{n} \lambda_i z_i^2,$$

$$0 \leqslant V^k - V^* \leqslant \tilde{x}^T R_k \tilde{x} = \sum_{i=k+2}^{n} (\lambda_i - \lambda_{k+1}) z_i^2.$$

Consequently,

$$\frac{V^k - V^*}{V_k} \leqslant \frac{\sum_{i=k+2}^{n} (\lambda_i - \lambda_{k+1}) z_i^2}{\sum_{i=1}^{n} \lambda_i z_i^2} \leqslant \frac{\sum_{i=k+2}^{n} (\lambda_i - \lambda_{k+1}) z_i^2}{\sum_{i=k+2}^{n} \lambda_i z_i^2}.$$

Since $\frac{\sum_{i=k+2}^{n} (\lambda_i - \lambda_{k+1}) z_i^2}{\sum_{i=k+2}^{n} \lambda_i z_i^2} \leqslant 1$ and $\sum_{i=k+2}^{n} (\lambda_n - \lambda_i) z_i^2 \geqslant 0$, we have

$$\begin{aligned}\frac{\sum_{i=k+2}^{n} (\lambda_i - \lambda_{k+1}) z_i^2}{\sum_{i=k+2}^{n} \lambda_i z_i^2} &\leqslant \frac{\sum_{i=k+2}^{n} (\lambda_i - \lambda_{k+1}) z_i^2 + \sum_{i=k+2}^{n} (\lambda_n - \lambda_i) z_i^2}{\sum_{i=k+2}^{n} \lambda_i z_i^2 + \sum_{i=k+2}^{n} (\lambda_n - \lambda_i) z_i^2}\\ &= \frac{\sum_{i=k+2}^{n} (\lambda_n - \lambda_{k+1}) z_i^2}{\sum_{i=k+2}^{n} \lambda_n z_i^2}\\ &= \frac{\lambda_n - \lambda_{k+1}}{\lambda_n}.\end{aligned}$$

Therefore $\frac{V^k - V^*}{V_k} \leqslant \frac{\lambda_n - \lambda_{k+1}}{\lambda_n}$ and $\frac{V^k}{V^*} \leqslant \frac{\lambda_n}{\lambda_{k+1}}$. □

For the case that the eigenvalues $\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_n$ centralize to a small interval, and $\lambda_1, \lambda_2, \ldots, \lambda_k$ distribute in a broad interval, the approximation solution performs well.

Table 1 Original lower bound γ_k and improved lower bound $\gamma_k + d_k$ for $k = 0, 1, \ldots, 9$

k	0	1	2	3	4	5	6	7	8	9
γ_k	–516.7	–506.8	–472.7	–449.7	–432.7	–428.4	–427.6	–424.3	–422.7	–422
$\gamma_k + d_k$	–438.7	–447.9	–449.0	–449.0	–432.6	–428.4	–427.6	–424.3	–422.7	–422

5 Numerical example

Now we give an example to illustrate our approach.

Example 2 We consider a randomized instance of (QIP) problem with

$$Q = \begin{bmatrix} 19 & 19 & -14 & 12 & -4 & 18 & -7 & -5 & 19 & 0 \\ 19 & 6 & -7 & 1 & -3 & -6 & 20 & 21 & 9 & -25 \\ -14 & -7 & 30 & 0 & 2 & 7 & 0 & 1 & 16 & -33 \\ 12 & 1 & 0 & 23 & -14 & -24 & -13 & -5 & 2 & -1 \\ -4 & -3 & 2 & -14 & 35 & -19 & -3 & -17 & 18 & -6 \\ 18 & -6 & 7 & -24 & -19 & 26 & -21 & -15 & -2 & -2 \\ -7 & 20 & 0 & -13 & -3 & -21 & 17 & -17 & -16 & 0 \\ -5 & 21 & 1 & -5 & -17 & -15 & -17 & 12 & 15 & 9 \\ 19 & 9 & 16 & 2 & 18 & -2 & -16 & 15 & -27 & -2 \\ 0 & -25 & -33 & -1 & -6 & -2 & 0 & 9 & -2 & 17 \end{bmatrix}.$$

An optimal solution is

$$x^* = [-1, -1, -1, 1, 1, 1, 1, 1, 1, -1]^T,$$

with optimal value $\gamma = -422$.

To solve the low rank problem $(\mathcal{P}_1)$, we have an approximation solution

$$\tilde{x} = [-1, -1, -1, 1, -1, 1, 1, 1, 1, -1]^T,$$

with an objective value of $\tilde{x}^T Q \tilde{x} = -326$. The difference between these two values is 96, and the evaluated bound $\Delta_1 = \tilde{x}^T R_1 \tilde{x} - d_1$ discussed in Lemma 6 is 121.91, which is larger than the real error.

Solving $(\mathcal{P}_2)$, we obtain an approximation solution x^*, which is an optimal solution of (QIP). But $\gamma_2 = -472.70$, $d_2 = 23.66$ and the error bound $\Delta_2 = \tilde{x}^T R_2 \tilde{x} - d_2$ is 27.04, which is not zero.

For this instance, the lower bounds γ_k and their improved lower bounds $\gamma_k + d_k$ are shown in Table 1, and plotted in Fig. 2.

In Fig. 2, the lower bound γ_k gets tighter as k increases, and the improved lower bound $\gamma_k + d_k$ is tighter than γ_k, but does not get the optimal value for $k < 9$.

6 Conclusions

An improved lower bound method by incorporating an ignored tail term in the original method discussed in [15] is proposed. Our proofs are based mainly on spectral decomposition of the quadratic matrix in the objective function. At the same time, an approximation algorithm for the binary quadratic programming problem is naturally developed and the worst case performance analysis is applied.

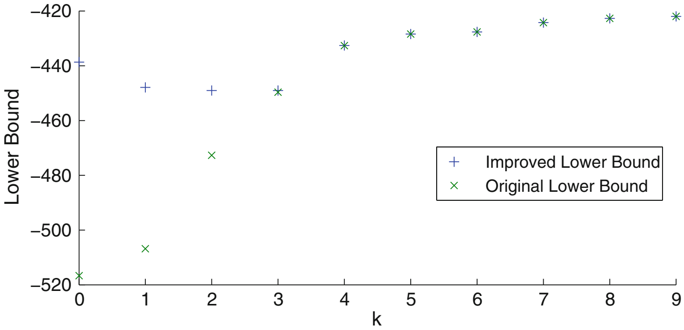

Fig. 2 Original lower bound γ_k and improved lower bounds $\gamma_k + d_k$ for $k = 0, 1, \ldots, 9$

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